Compact Binary Merger Gravitational Wave (GW) Signal Model for a Rotating Earth

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This research was supported by NSF Physics Frontier Center

Project Overview

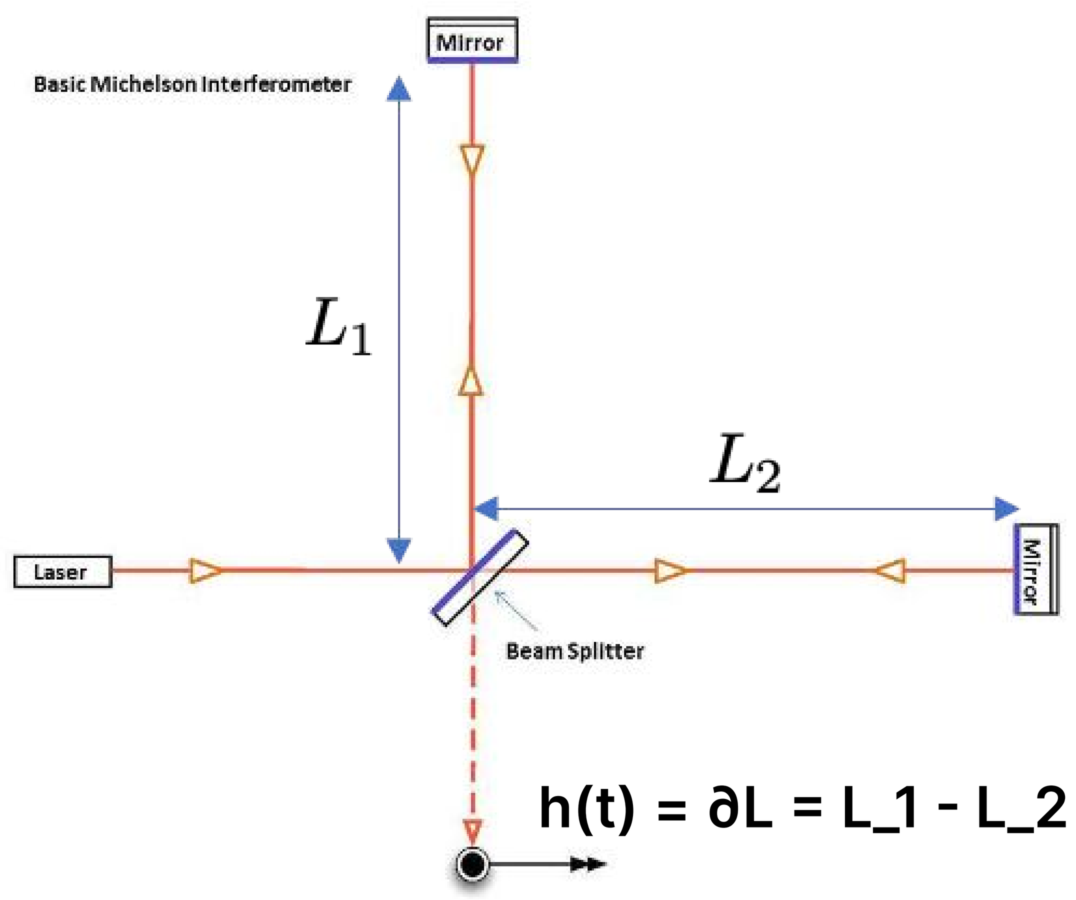

Figure 1: LIGO Livingston, LIGO Hanford, and VIRGO Interferometries setup. h(t) is the time series measured and recorded. ligo.caltech.edu

- At its most sensitive state, LIGO is able to detect a change in distance between its mirrors 1/10,000th the width of a proton!
- Streams of length differential data are noise-reduced (high pass filter) and masked to search for black hole and neutron-star merger events, which produce GW signals.
- The axis orientation of each detector determines its receptivity to the + ("plus") and x ("cross") GW polarizations, indicated by detector-specific quantities F_+ and F_x, respectively.

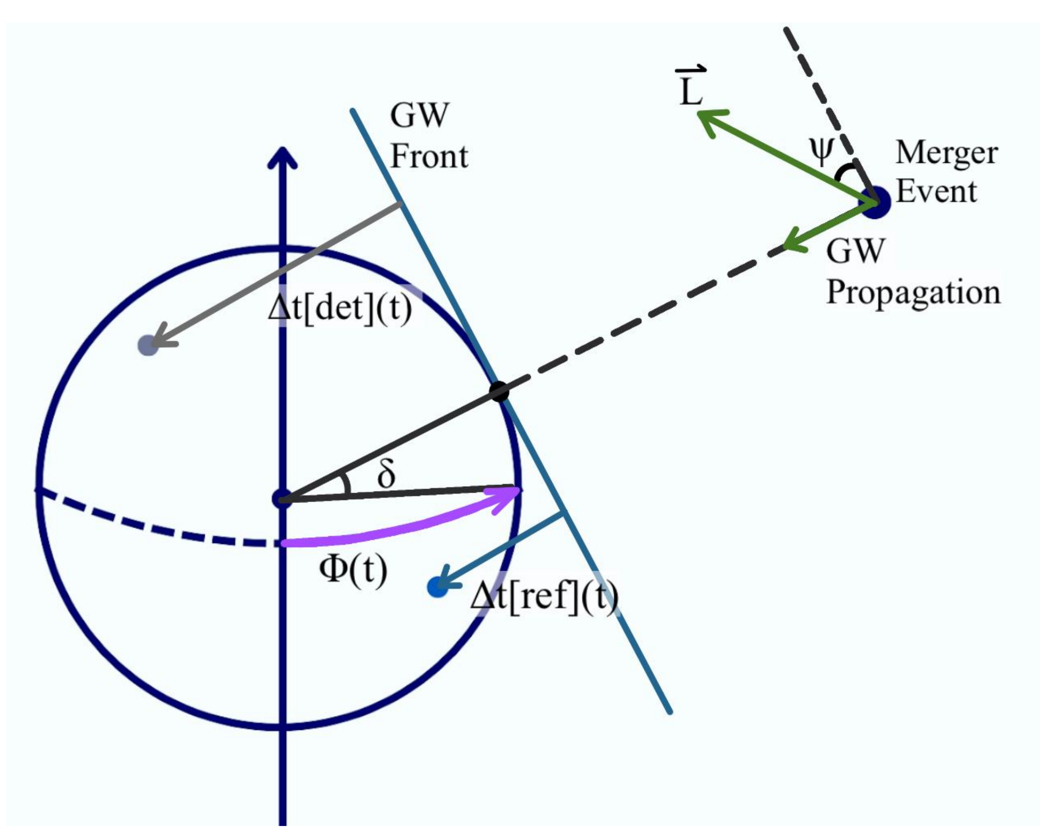

Figure 2: Visualizes the receipt of GW signals on Earth. ø(t) and ∂ are determined by the sky location of the event, ψ is determined by the merger angular momentum.

$$h(t) = F_+[det] \cdot h_+(t - \Delta t[det]) + F_x[det] \cdot h_x(t - \Delta t[det])$$

$$h_{rot}(t) = F_+[det](t) \cdot h_+(t - \Delta t[det](t)) + F_x[det](t) \cdot h_x(t - \Delta t[det](t))$$

Equations 1, 2: GW model in non-rotating Earth and rotating Earth. F_+, F_x, and Δt are given time-dependence.

- ø(t) = ø_0 + Ωt, where Ω is the rate of rotation of the Earth. In the non-rotating Earth model, Ω = 0. That is, ø(t) = ø_0.
- F_+ and F_x are dependent on ø(t), ∂, and ψ. *Δt is dependent on ø(t) and ∂.* Because of ø, F_+, F_x, and Δt are time-dependent in the rotating Earth model.

GOAL: Using perturbative expansion in Ω to determine a closed-form expression for h_rot in the frequency domain.

Linear Perturbation in Ω and Model Evaluation

$$F_+(t) = F_+(t_0) + \Omega \cdot (t - t_0) \cdot \gamma_+$$

$$F_x(t) = F_x(t_0) + \Omega \cdot (t - t_0) \cdot \gamma_x$$

$$\Delta t(t) = \Delta t(t_0) + \Omega \cdot (t - t_0) \cdot \eta$$

Equations 3, 4, 5: Our newly time-dependent terms are expressed to be linear in Ωt. Higher order terms in Ω do exist, but they are eliminated. F_+(t_0), F_x(t_0), and Δt(t_0) are used in the non-rotating Earth model.

SIDE NOTE: Closed form expressions for the constant terms γ_+, γ_x, and η, have been calculated in terms of previously defined ø_0, ∂, and Ψ.

$$h_{rot}(t) - h(t) = \Omega \cdot (t - t_0) \cdot \gamma_+ \cdot h_+(t - \Delta t(t_0)) - F_+(t_0) \cdot \Omega \cdot (t - t_0) \cdot \eta \cdot h'_+(t - \Delta t(t_0)) + \Omega \cdot (t - t_0) \cdot \gamma_x \cdot h_x(t - \Delta t(t_0)) - F_x(t_0) \cdot \Omega \cdot (t - t_0) \cdot \eta \cdot h'_x(t - \Delta t(t_0))$$

Equation 6: Broadening our first-order-Ω approximation to h_rot, we can extract the model we already have, h(t), leaving us with first-order-Ω(t-t0) terms.*

- It is known how to compute the model for GW signal polarizations given a set of parameters. That is, we know h_+ and h_x in both the time and frequency domains given a set of parameters.
- Derivatives, in particular, h_+'(f) and h_x'(f), can be computed via finite-differencing in python.

$$t \cdot x(t) \xrightarrow{F} \frac{i}{2\pi} \frac{\partial}{\partial f} \tilde{x}(f)$$

$$x'(t) \xrightarrow{F} 2\pi i f \cdot \tilde{x}(f)$$

$$x(t - \tau) \xrightarrow{F} \tilde{x}(f) \cdot \exp(-2\pi i f \cdot \tau)$$

Equations 7, 8, 9: A few key Fourier tricks used to solve this problem are shown.

OBSERVATION: All terms in Equation 6 consist of some combination of the transformations demonstrated in Equations 7, 8, 9 on h_+(t) and h_x(t), as well as constant coefficient multiplication.

$$\alpha_0 = \exp(-2\pi i f \cdot \Delta t(t_0))$$

$$\alpha_1^+ = \alpha_0 \cdot \{\eta \cdot \Omega \cdot F_+(t_0) \cdot [1 - 2\pi i f \cdot (\Delta t(t_0) - t_0)] + F_+(t_0) + \gamma_+ \cdot \Omega \cdot [\Delta t(t_0) - t_0]\}$$

$$\alpha_2^+ = \alpha_0 \cdot \Omega \cdot \left\{\gamma_+ \cdot \frac{i}{2\pi} + \eta \cdot F_+(t_0) \cdot f\right\}$$

Equations 10, 11, 12: A few constants are defined to condense the following significant result. Note that α_1x and α_2x can be inferred from α_1+ and α_2+ above by replacing occurances of '+' with 'x'.

$$\tilde{h}_{rot}(f) = \alpha_1^+ \cdot \tilde{h}_+(f) + \alpha_2^+ \cdot \tilde{h}'_+(f) + \alpha_1^x \cdot \tilde{h}_x(f) + \alpha_2^x \cdot \tilde{h}'_x(f)$$

Equation 13: From our observation above, and a lot of math, we can compute the Fourier transform of h_rot(t).

RESULT: We have a first order (in Ω) approximation for h_rot in the frequency domain. Notably, we can evaluate ȟ_rot(f) at a particular frequency from a set of model parameters.

Next Steps and Applications

$$P(\theta|d) = \frac{P(d|\theta)\,P(\theta)}{P(d)}$$

$$P(\theta \mid d) \propto P(d \mid \theta) \cdot P(\theta) = \mathcal{L} \cdot \text{Prior}$$

$$log\mathcal{L} = \langle d(f)|h(f)\rangle - \tfrac{1}{2}\langle d(f)|h(f)\rangle$$

Equation X: Bayes's Theorem. Equation Y: Discarding d-dependent factors. Equation Z: log likelihood, assuming noise is gaussian-distributed in the frequency domain.

- θ, merger parameters: masses, spins, location, orientation, tidal deformity (neutron star mergers).
- d(f), data: filtered time snippet of data isolated by masking sample of 100 GW signal images.

$$\langle a|b\rangle = \text{Re}\left\{\sum_f \frac{a^*(f)b(f)}{\frac{T}{2} \cdot S_n(f)}\right\}$$

Equation A: Inner product defined. S_n(f) represents spectral noise density in the frequency domain.

APPLICATION: Forming a posterior distribution of the data in parameter space via Monte Carlo Markov Chain Simulations or Fischer Analysis.

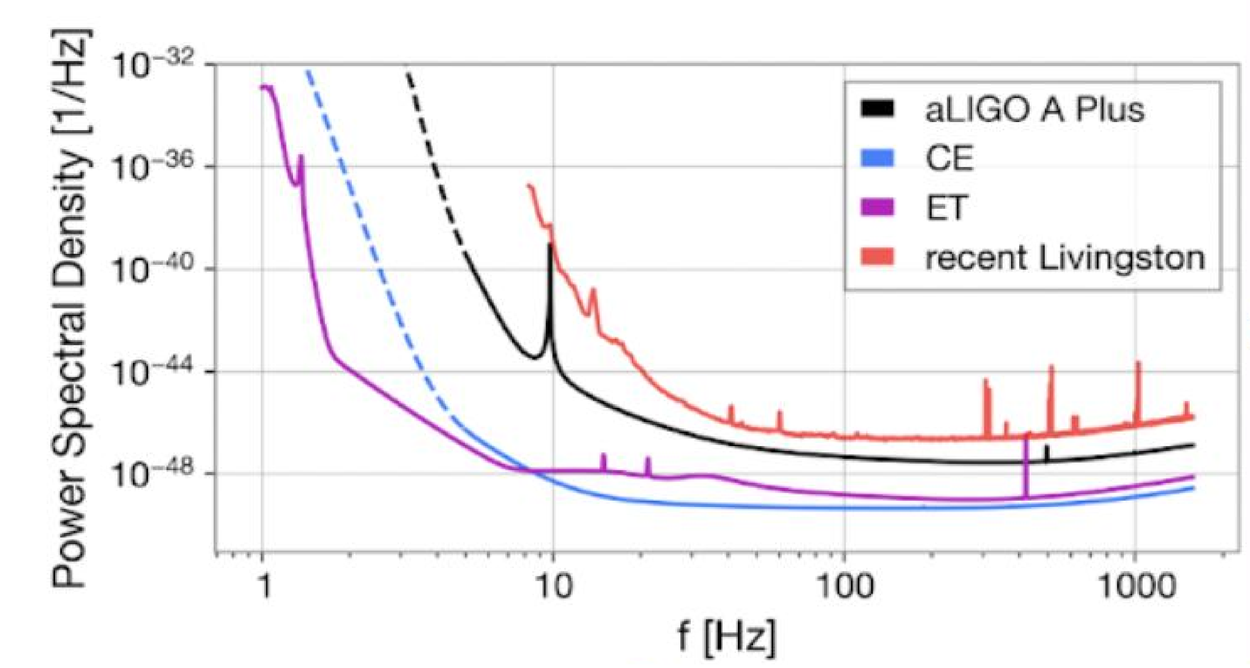

Figure X: The PSDs of Upgraded LIGO, Cosmic Explorer, Einstein Telescope, and LIGO Livingston are compared. Lazarow-Leslie-Dai 2024.

- Current frequency cutoff: 8 Hz.
- Smaller net amplitude of low-frequency noise in up-and-coming detectors will allow for a wider frequency range and a longer discernible signal duration. Necessitates the rotating Earth model.
- Some model modifications will be needed for the modified experimental setup of ET.

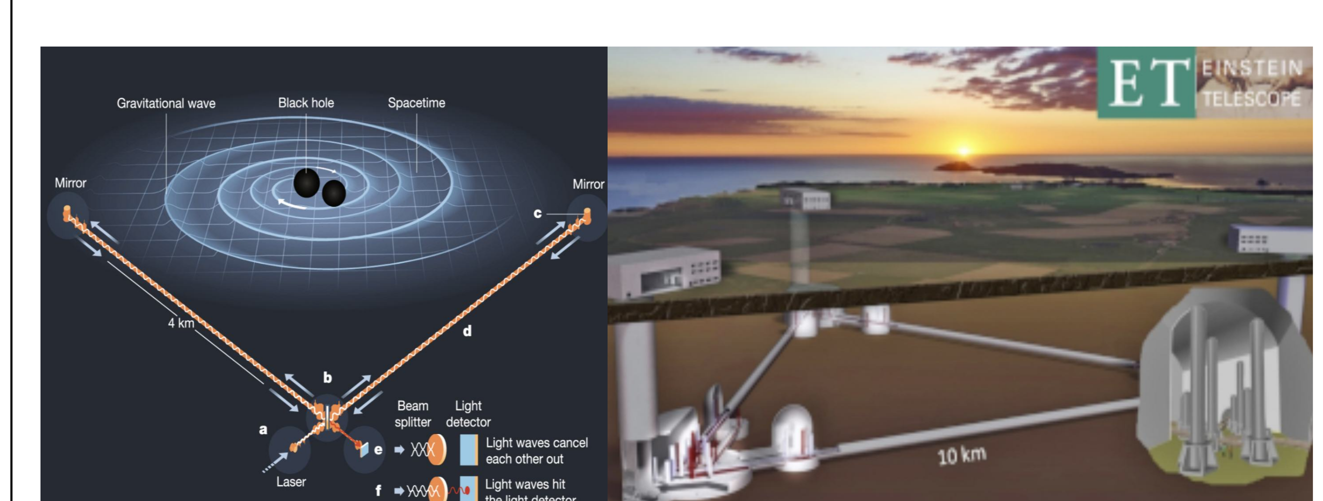

Figure X: The experimental models of Cosmic Explorer (left) and the Einstein Telescope (right) are shown.